Several Outcomes on Prime Labeling of Families of Graphs

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Abstract: In the present work we investigate some classes of graphs and disjoint union of some classes of graphs which admit prime labeling. We also investigate prime labeling of a graph obtained by identifying two vertices of two graphs. We also investigate prime labeling of a graph obtained by identifying two edges of two graphs. Prime labeling of a prism graph is also discussed. We show that a wheel graph of odd order is switching invariant. A necessary and sufficient condition for the complement of W_n to be a prime graph is investigated.

Keywords: Graph Labeling, Prime Labeling, Switching of a Vertex, Switching Invariance

Introduction:
We begin with simple, finite, undirected and non-trivial graph G = (V, E) with the vertex set V and the edge set E. The number of elements of V denoted as |V| is called the order of the graph G while the number of elements of E, denoted as |E| is called the size of the graph G. In the present work C_n denotes the cycle with n vertices and P_n denotes the path of n vertices. In the wheel W_n = C_n + K_1 the vertex corresponding to K_1 is called the apex vertex and the vertices corresponding to C_n are called the rim vertices. For various graph theoretic notations and terminology we follow Gross and Yellen [1] whereas for number theory we follow D. M. Burton [2]. We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1: If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

For latest survey on graph labeling we refer to J. A. Gallian [3]. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades. For any graph labeling problem following three features are really noteworthy:

- a set of numbers from which vertex labels are chosen;
- a rule that assigns a value to each edge;
- a condition that these values must satisfy

The present work is aimed to discuss one such labeling known as prime labeling.

Definition 1.2: A prime labeling of a graph G of order n is an injective function f : V → {1, 2, ..., n} such that for every pair of adjacent vertices u and v, gcd (f(u), f(v)) = 1. The graph which admits prime labeling is called a prime graph.

Definition 1.3: A vertex switching G_v of a graph G is the graph obtained by taking a vertex G, removing all the edges incident to v and adding edges joining to every other vertex which is not adjacent to v in G.

Definition 1.4: A prime graph is said to be switching invariant if for every vertex v of G the graph G obtained by switching the vertex v in G is also a prime graph.

Definition 1.5: For two graphs G_1 = (V_1, E_1) and G_2 = (V_2, E_2) their
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Cartesian product $G_1 \times G_2$ is defined as the graph whose vertex set is $V_1 \times V_2$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ in $G_1 \times G_2$ are adjacent if $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ or $u_1$ is adjacent to $u_2$ and $v_1 = v_2$.

**Definition 1.6:** $C_n \times P_2$ is called prism graph.

**Bertrand's Postulate 1.7:** For every positive integer $n > 1$ there is a prime $p$ such that $n < p < 2n$.

**2. Some Results on Prime Labeling:**

**Theorem 2.1:** The complement of $W_n$ is prime if and only if $3 \leq n \leq 6$.

**Proof:** We can easily see that $W_n$ for $n = 3, 4, 5$ and $6$ from figure 1. Now if $n \geq 7$ then $(n - 3) \geq 4$ and every rim vertex of $W_n$ is adjacent to other $(n - 3)$ rim vertices. We have total $\frac{n+1}{2}$ even numbers to assign $n + 1$ vertices. If one of the rim vertices is labeled as even number then other $n - 3$ vertices cannot be labeled as even number. Also remaining two rim vertices are adjacent, as only one of them can be labeled as even number. The apex vertex can also be labeled as even number. Thus maximum three vertices can be labeled as even number. But if $n \geq 7$ then we have 4 or more even numbers to label. So it is not possible. Thus $W_n$ is not prime for $n \geq 7$.

![Figure 1. Prime labeling of $W_3, W_4, W_5$ and $W_6$](image)

**Theorem 2.2:** The graph obtained by switching a rim vertex in $W_7$ is not a prime graph.

**Proof:** Let $v_1, v_2, v_7$ be consecutive rim vertices of $W_7$ and $v_0$ be the apex vertex of $W_7$. Let $G$ be the graph obtained by switching the vertex $v_7$. If possible let $f : V(G) \rightarrow \{1, 2, 3, 4, 5, 6\}$ be a prime labeling. As $v_7$ is adjacent to four vertices $v_2, v_3, v_4, v_5$ and $v_0$ is adjacent to six vertices $v_1, v_2, v_3, v_4, v_5, v_6$, the labels of $v_7$ and $v_0$ cannot be even. Moreover we have to distribute four even labels among six vertices. Therefore at least two adjacent vertices from $v_1, v_2, v_3, v_4, v_5, v_6$ will receive the even labels which contradicts the fact that $f$ is a prime labeling. We noticed that it is not easy to discuss the prime labeling of a graph obtained by switching any rim vertex of $W_n$ when $n + 1$ is a composite number.

**Theorem 2.3:** If $n \geq 3$ is an odd integer then the prism graph $C_n \times P_2$ is not prime.

**Proof:** In the prism graph $C_n \times P_2$ there are two cycles $C_n$. So total numbers of vertices are $2n$. So we have to use $1$ to $2n$ natural numbers to label these vertices and from $1$ to $2n$ there are $n$ even integers. If $n$ is odd then we can use at most $\frac{n-1}{2}$ even integers to label the vertices of a cycle $C_n$. We have such two cycles, so we use at most $\frac{n-1}{2} + \frac{n-1}{2} = n-1$ even integers to label the vertices of $C_n \times P_2$. But from $1$ to $2n$ there are $n$ even integers. So such prime labeling is not possible.
Thus $C_n \times P_2$ is not prime if $n \geq 3$ is an odd integer.

3. Concluding Remark:
Study of relatively prime numbers is very interesting in the theory of numbers and it is challenging to investigate prime labeling of some families of graphs. Here we investigate several results of some classes of graphs about prime labeling. Extending the study to other graph families is an open area of research.

References: